How many Handshakes?

**National Council of Teachers of Mathematics Standards Grades 6-8**
(summarized - a complete list can be found in the lesson plan linked above):

**Numbers and Operations:**
- Develop an understanding of large numbers and recognize and appropriately use exponential, scientific, and calculator notation.
- Understand the meaning and effects of arithmetic operations with fractions, decimals, and integers.

**Algebra:**
- Identify functions as linear or nonlinear and contrast their properties from tables, graphs, or equations.
- Develop an initial conceptual understanding of different uses of variables.

**Geometry:**
- Use visual tools such as networks to represent and solve problems.
- Recognize and apply geometric ideas and relationships in areas outside the mathematics classroom, such as art, science, and everyday life.

**Problem Solving:**
- Build new mathematical knowledge through problem solving.
- Apply and adapt a variety of appropriate strategies to solve problems.

**Reasoning and Proof:**
- Recognize reasoning and proof as fundamental aspects of mathematics.
- Develop and evaluate mathematical arguments and proofs.

**Communication:**
- Organize and consolidate mathematical thinking through communication.
- Use the language of mathematics to express mathematical ideas precisely.

**Connections:**
- Recognize and use connections among math ideas.
- Recognize and apply mathematics in contexts outside mathematics.

**Representation:**
- Create and use representations to organize, record, and communicate mathematical ideas.
- Select, apply and translate among mathematical representations to solve problems.
Ohio Department of Education Academic Content Standards: Mathematics Grades 5-7

Patterns, Functions and Algebra Standard
- Use rules and variables to describe patterns, functions, and other relationships.
- Use symbolic algebra to represent and explain mathematical relationships.
- Write, simplify, and evaluate algebraic expressions.

Mathematical Processes Standard
- Relate mathematical ideas to one another and to other content areas.
- Communicate mathematical thinking to others and analyze the mathematical thinking and strategies of others.
- Use representations to organize and communicate mathematical thinking and problem solving.

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Description
In this lesson, students begin looking at a simple counting problem, use a line graph to display the data, and extend the pattern to algebraic thinking. Simple information is organized and analyzed before it is extended to make predictions. Using the organizational tools, students will investigate for patterns. These patterns can be written mathematically using algebraic thinking to solve much larger numbers of data. This problem-solving strategy of solving a simple problem and extending the thinking to a more complex problem is a useful mathematical skill in processing information.

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Objectives
1. The students will investigate and solve a non-routine problem.
2. In the context of solving a non-routine problem, students will organize data in a chart and display it in a line graph.
3. Students will explore the information found and make a conjecture about the pattern and use the line graph to make a table of predictions.
4. Students will explain the pattern in detail using the information found and evaluating the predictions made and extend the pattern beyond the one given.

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Materials
1. Student worksheets
2. Calculators
**Procedures**
This is an abbreviated list; please link to the plan and worksheets for further detail.

1. The class will discuss the problem on the class lesson sheet and the desired results.
2. Each student prepares a reflective essay predicting what the outcome will be. This should support the student's estimated guess with some simple facts that can be determined by counting.
3. The class will be divided into groups of four students. Each group will investigate the total number of handshakes that occur in small groups by counting, by drawing a diagram showing the handshakes, and how each total was arrived at for two, three, four, and five people.
4. Each group creates a chart of the number of people and the total number of handshakes that would occur if each person shook hands with everyone else only once for up to eleven people.
5. Each group should summarize its results by writing about the pattern generated.
6. Within the groups, the students use the column of ordered pairs in the data chart to graph the information in the first quadrant of the coordinate plane. Use a solid line to connect the points and a dotted line to predict where the line would continue to be for 15, 18, and 20 people.
7. Students should consider the dotted line and make a table of predicted outcomes and use the discovered pattern to determine the actual number of handshakes.
8. Sharing to the class, the groups should explain the pattern in detail. Individually, each student should prepare and extended response essay discussing the patterns discovered and evaluations of the predictions made.
9. Each student's knowledge will be assessed to see how their conjectures fit for numbers 50, 100, and 250.
Lesson Plan

The Handshake Problem:

In this room, shake everyone’s hand once. How many total handshakes will there be? Is there a way to determine how many handshakes there would be if the number of people in the room was 50? 100? Any number of people?

Quick Key for Teacher Guidance

The addition pattern is \((n - 1) + (n - 2) + (n - 3) + \ldots + (n - n)\). This leads to the expression \(\frac{1}{2}(n)(n – 1)\) which simplifies to \(\frac{n^2 - n}{2}\).

Big Idea of this Lesson:

Students begin looking at a simple counting problem, use a line graph to display the data, and extend the pattern to algebraic thinking. Simple information is organized and analyzed before it is extended to make predictions. Using the organizational tools, students will investigate for patterns. These patterns can be written mathematically using algebraic thinking to solve much larger numbers of data. This problem-solving strategy of solving a simple problem and extending the thinking to a more complex problem is a useful mathematical skill in processing information.

Specific Learning Objective: Grades 6 – 9  Time Frame: 120 minutes

Objectives

1. The students will investigate and solve a non-routine problem.
2. In the context of solving a non-routine problem, students will organize data in a chart and display it in a line graph.
3. Students will explore the information found and make a conjecture about the pattern and use the line graph to make a table of predictions.

Students will explain the pattern in detail using the information found and evaluating the predictions made and extend the pattern beyond the one given.

NCTM Standards for Grades 6 - 8:

1. Numbers and Operations:
   - Develop an understanding of large numbers and recognize and appropriately use exponential, scientific, and calculator notation.
• Develop meaning for integers and represent and compare quantities with them.
• Understand the meaning and effects of arithmetic operations with fractions, decimals, and integers.
• Use the associative and commutative properties of addition and multiplication and the distributive property of multiplication over addition to simplify computations with integers, fractions, and decimals.
• Understand and use the inverse relationships of addition and subtraction, multiplication and division, and squaring and finding square roots to simplify computations and solve problems.
• Select appropriate methods and tools for computing with fractions and decimals from among mental computation, estimation, calculators or computers, and paper and pencil, depending on the situation, and apply the selected methods.
• Develop and analyze algorithms for computing with fractions, decimals, and integers and develop fluency in their use.
• Develop and use strategies to estimate the results of rational-number computations and judge the reasonableness of the results.

2. Algebra
• Represent, analyze, and generate a variety of patterns with tables, graphs, words, and, when possible, symbolic rules.
• Relate and compare different forms of representations for a relationship.
• Identify functions as linear or nonlinear and contrast their properties form tables, graphs, or equations.
• Develop an initial conceptual understanding of different uses of variables.
• Explore relationships between symbolic expressions and graphs of lines, paying particular attention to the meaning of intercept and slope.
• Use symbolic algebra to represent situations and to solve problems, especially those that involve linear relationships.
• Recognize and generate equivalent forms for simple algebraic expressions and solve linear equations.
• Model and solve contextualized problems using various representations, such as graphs, tables, and equations.
• Use graphs to analyze the nature of changes in quantities in linear relationships.

3. Geometry
• Use visual tools such as networks to represent and solve problems.
• Use geometric models to represent and explain numerical and algebraic relationships.
• Recognize and apply geometric ideas and relationships in areas outside the mathematics classroom, such as art, science, and everyday life.

4. Problem Solving:
• Build new mathematical knowledge through problem solving.
• Solve problems that arise in mathematics and in other contexts.
• Apply and adapt a variety of appropriate strategies to solve problems.
• Monitor and reflect on the process of mathematical problem solving.
5. **Reasoning and Proof:**
   - Recognize reasoning and proof as fundamental aspects of mathematics.
   - Make and investigate mathematical conjectures.
   - Develop and evaluate mathematical arguments and proofs.
   - Select and use various types of reasoning and methods of proof.

6. **Communication:**
   - Organize and consolidate their mathematical thinking through communication.
   - Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
   - Analyze and evaluate the mathematical thinking and strategies of others.
   - Use the language of mathematics to express mathematical ideas precisely.

7. **Connections:**
   - Recognize and use connections among mathematical ideas.
   - Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
   - Recognize and apply mathematics in context outside of mathematics.

8. **Representation:**
   - Create and use representations to organize, record, and communicate mathematical ideas.
   - Select, apply, and translate among mathematical representations to solve problems.
   - Use representations to model and interpret physical, social and mathematical phenomena.

**Ohio Department of Education Academic Content Standards: Mathematics Grades 5-7**

**Patterns, Functions and Algebra Standard**
- Use rules and variables to describe patterns, functions, and other relationships.
- Use symbolic algebra to represent and explain mathematical relationships.
- Write, simplify, and evaluate algebraic expressions.

**Mathematical Processes Standard**
- Relate mathematical ideas to one another and to other content areas.
- Communicate mathematical thinking to others and analyze the mathematical thinking and strategies of others.
- Use representations to organize and communicate mathematical thinking and problem solving.

**Methods:**

Lesson Plan for NEOCEX and ORC KuBoyle, CSU
1. Read the story on the Class Lesson Sheets about the problem. Discuss the situation and desired results.
2. Each student is to write a reflective essay predicting what the outcome will be. The essay is to support the student’s estimated guess with some simple facts that can be determined by counting.
3. Assign students to groups of four. Each group is to investigate the total number of handshakes that occur in small groups by counting by drawing a diagram showing the handshakes and how each total was arrived at for two, three, four, and five people.
4. Each group is to create a chart of the number of people and the total number of handshakes that would occur if each person shook hands with everyone else only once for up to eleven people.
5. Assess each student on the results of the investigation by writing about the pattern.
6. As the assigned group once again, have the students use the column of ordered pairs in the data chart to graph the information in the first quadrant of the coordinate plane. Use a solid line to connect your points. Use a dotted line to predict where you think the line would continue to be for fifteen, eighteen, and twenty people.
7. Looking at the dotted line extending the graph and make a table of predicted outcomes. Then, using the discovered pattern, determine the actual number of handshakes. Share this information with the class.
8. Explain the pattern in detail using the discovered information and evaluating the predictions made by writing an extended response essay.
9. Assess each student’s knowledge through an extended response essay that illustrates how the total number of handshakes for any given number of people can be determined. Evaluate this conjecture for the numbers 50, 100, and 250.

Management Issues:

The author’s comments: Some students are quick to start the problem by counting but get stuck immediately at how to record what is happening. Other students don’t mind doing this with just a few people to model the situation, but complain when the number of people gets over ten. They cannot see the value of looking for a pattern.

When students are making conjectures, many students just want to guess. The difficulty lies in getting them to organize the information into the data charts in order to determine what is happening. Making then testing a conjecture is new to them and they are amazed at the persistence it requires. They tend to want to give up immediately and say that this is just too hard to do. Along with statements about try and try again, I found that a conversation about networking was a useful tool to move them in the right direction.

Materials/Technology:

Class Lesson Sheets and Student Worksheets
Calculators
Misconceptions:

Students are very capable of determining the addition pattern after they make a good network diagram. Some students needed the hint of drawing the handshaking participants in a circular pattern instead of a linear pattern.

The author’s experience: Going from the addition pattern to the symbolic was more difficult. I suggested to some students a simpler problem by just looking at what happens to 4, 5, 6, and 7 people. Then extrapolate from there. I had the students represent the handshakes in columns with numbers and symbols encouraged the addition and simplification process. This helped. Simplifying the expression was a real problem as the students saw the ratio as 2 to 1 instead of 1 to \( \frac{1}{2} \) for the binomial. In other words, they saw the first number, \( n(n-1) \) as twice as large as the second number \( \frac{1}{2}(n)(n-1) \) and wanted to solve the expression for the first number (\( n \) squared – \( n \)) only making the solution to the handshake problem double what it should be. They fudged the answer at this stage and had no idea what went wrong.

Assessment:

See the Student Worksheets.

(Teaching tip: Copy the worksheets with one question to a page. As students answer the question, collect the paper and rubric grade it. When the project is finished, pass back all the papers for the students to collate and compute their own final grade. As they do this, have them write about the activity as a whole and the fairness of this final result.)

Applications:

An understanding of networking is important for middle school students as it develops good number reasoning abilities and good communication skills. This problem is a classic in number theory for the results show math process in action.

Research Base/References:

Classic number theory problem

Equity:

Groups need to be assigned by the teacher as simply a group not as cooperative learning groups.

Many students will get frustrated by the process if the situation is not acted out a number of times before they get to work on the patterns.
Students with special needs can be encouraged to continually check the process with the calculator. An aide may be necessary to help record the thinking process.